PERCEPTION

&

CONTROL

OF

SELF-MOTION

edited by

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Chapter 22

Reciprocities of intentional systems

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22.1   Introduction

22.1.1   Dynamics of intention

Our aim is to generalize certain methods used in the study of dynamical systems to living systems whose behaviors express the intentional selection of goals. Goal-directed movements are as commonplace as any other form of motion studied by science, yet we seem to know so much less about them. But there is no reason to assume that dynamical principles apply only to the aimless behavior of nonliving systems. If nature is truly economical, then the shared evolutionary lineage of living systems with their near inanimate prototypes (e.g., viruses and DNA) makes likely a continuing reign of dynamical principles common both to physical laws and biological constraints.

Could the evolution of mind escape sharing these same foundations? We assume not. A thwart to this approach is the extreme complexity of systems capable of intentional behaviors.

Section 22.2 characterizes an intentional system as a mathematically simpler object than it really is—a group whose properties might be studied in lieu of the hidden workings of mind, much in the way that physicists study the group properties of sub-atomic particles. Physicists have found it extremely useful to express in group form the invariant properties shared by a class of complex systems. Psychologists might also, where the group properties embody the reciprocity mappings between animal and environment that sustain goal-directed behaviors. The group of complex involutions is our best guess as to the minimal structure that an intentional system must have. Groups are like yardsticks or other measuring devices revealing certain property dimensions while blatantly ignoring others. A group representation is not the object of study, nor even a model of it, any more than a yardstick or thermometer is a model of the phenomena they measure.

What should psychologists expect to gain from the study of dynamical principles? One possibility is that intentional dynamics, that is, the dynamics of living systems driven by intentions is, up to a point, governed by linear principles. By *linear* principles we mean those that explain why the response of a system is predictable from its history of inputs given certain boundary conditions. The relevant boundary conditions might well be the satisfying of certain intentions (construed as choices of goals and the means for seeking them).

Section 22.3 discusses and applies some of the principles often used to study the linear aspects of dynamical systems. Of special importance are Lie groups with a bracket operator (i.e., Lie algebras). This abstraction allows us to associate the perceiving-acting cycle with a manifold on which information- and energy-flows between and within interior (organismic) and exterior (environmental) frames are treated as components of a more abstract flow—a flow that is conserved if and only if an intended action is successfully completed. To prepare for this conclusion we work toward a Lie algebra representation for the perceiving-acting cycle as a way to mathematically motivate a generalized Hamiltonian description of intentional dynamics. A key result is an argument showing how intention acts as an invariant of motion for intentional systems analogous to invariants of motion that classical mechanics has identified for causal systems.

The aim of the chapter is to formally prescribe physically interpretable consequences of successful goal-directed activities. Success would pave the way for moving the study of intentional systems from philosophy to science. A caveat is in order, however. Although we believe a linear theory is useful, we further believe a more general nonlinear theory is necessary.

22.1.2 Reciprocity of influence between animal and environment

The environment influences the organism and the organism influences the environment. This reciprocity of influence is made possible by the evolution of a complex interior of the organism—without which there would be only action and reaction as holds for inanimate systems. The central nervous system is a dynamic medium which maps kinematic and geometric descriptions arising from interactions with environmental force fields, back into environmental force fields through the controlled adjustments of force producing actuators. This is illustrated by the control of egomotion toward a goal, such as in the controlled application of manual forces on the force-stick of an aircraft in a terrain following task, or animals running over rough terrain to escape a predator.

The mappings sustained by the neural substrate are causal but not forceful. One part of the substrate changes the state of another in ways other than by mechanical push. Output is determined by input according to some lawful but low-energy rather than high energy-coupling. Such minimal couplings are coordinated by *informational mappings*, in the sense of *uncertainty reduction across states*. This reduction of uncertainty across the mappings is specific to the satisfaction of an intention (that is, the achievement of a goal-state). These mappings are possible because, while the kinetics and kinematics are tightly coupled between the organism and the environment, the central nervous system is a medium that is sensitive to changes in the properties of informative low energy fields (e.g., hormonal, pheromonal, acoustical, optical). Thus, the resulting changes in state are nearly always out of proportion to any forces associated with the process by which the properties are detected (i.e., the mapping is nonlinear with respect to global force laws).

The most fundamental question is what kind of mapping is the central nervous system performing in the service of an intention? An answer will be proposed in terms of the reciprocal mappings that govern self-motivated activities. Physical science provides many examples of systems that exhibit reciprocity of mutual influence by means of forceful connections with their environments. When expressed mathematically as reciprocal mappings, these examples provide theorems that might be generalized to influences by informational couplings. If so, then some aspects of the methods developed by physicists for studying reciprocal forceful influences might be adopted by psychologists for studying reciprocal informational influences. In this way, the breadth of causal analysis carried out in an exterior frame of reference might be extended to intentional (acausal) analysis which requires, in addition, an interior frame of reference.

22.2   Fundamental group of reciprocity maps for psychological inverse dynamics

22.2.1   Inverse dynamics

Classical inverse dynamics defines a special mapping between kinetic and kinematic parameters, where these mappings lie solely within a single exterior frame of reference. The interactions within this frame of reference take place between atomisms who are assumed to lack complex interiors. In short, there is no interior frame of reference in classical mechanics. This relationship is exemplified by Newton's law $F = ma$ or Hooke's law $F = kx$. Newton's law defines force relations on a kinetic field (under the boundary condition of constant mass), while Hooke's law defines force relations on a potential field (under the boundary condition of constant stiffness). In the former equation, the domain of the mapping is *force* (kinetic parameter) and the range of the mapping is *acceleration* (kinematic parameter). In the latter equation, the range is *displacement* (also a kinematic parameter). In both cases, the mapping from kinetic states to kinematic states is *one-to-one* from domain to range and *onto* from range to domain. This means the mapping has an *inverse*.[1]

The symmetry in this mapping has had profound implications both theoretically and experimentally. Many forces have hidden sources that are not easily measured (such as, those governing planetary motions, chemical interactions, hydrodynamic flows) but may be inferred from the motions they produce—in flows or maps—which can be more readily observed and measured. Our fundamental thesis is that the sources expressing intentional constraints, though also not readily available for measurement, may be specified by the behaviors produced in accordance with these constraints. If so, then there must be an "inverse dynamics" (although not of the classical sort) for which a symmetric mapping exists. What sort of mapping might that be?

In answering this question it is helpful to contrast three theories of how behavior might be "shaped" (Skinner), "cognized" (Tolman), or "tuned" (Gibson) toward a goal-state. These approaches differ in a fundamental way

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[1] A *mapping* is a correspondence from one set to another. It associates the elements in set A (domain) with elements in set B (range). Mappings may be one-to-many, many-to-one, or one-to-one; and they may be into or onto. If the direct mapping is one-to-one from A onto B, then the *inverse* mapping is obtained by reversing the direction of the correspondence, namely, from B onto A. As all the elements of B are images of the direct mapping from A, then all the elements of B are objects under the inverse mapping. The inverse mapping is one-to-one because the direct mapping is one-to-one. Similarly, because all the elements of A are objects under the direct mapping, then the inverse mapping is onto. There are no residual elements in inverse mappings.

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with respect to how the mapping is defined on frames of reference; namely, they differ in the number of frames of reference over which the mapping is defined. They also differ in how well they satisfy the requirements for an inverse dynamics.

22.2.2   Psychology and inverse dynamics

B.F. Skinner implicitly adopted a well-known method from classical physics in founding the method of operant conditioning. He cites Bertrand Russell as the source of the idea that the "reflex" in psychology has the same status as "force" in physics (Skinner, 1977). He admits to being influenced by Mach's (1919) *Science and Mechanics* wherein the method of inverse dynamics was championed. In operant conditioning, rather than associating stimuli with responses by means of reinforcement, one waits for the behavior to emerge and then reinforces it. In this approach the constraints retroact back *onto* the original behavior to shape it (measured operationally as the *change in the rate of responding*).

Like classical inverse dynamics, there is no room for contributions from an interior frame of reference. Systems are assumed to be without complex interiors—they are operationally defined atomisms acted on from the outside. Hence there is no provision for either the inflow of information or the outflow of intentional acts across the boundaries of the system: "For me the observable operations in conditioning, drive, and emotion lay *outside* the organism, but Tolman put them inside ..." (Skinner, 1977, p. 376).

Mathematically speaking, operant conditioning requires a mapping that is both commutative and invertible. These properties are illustrated in Figures 22.1a and 1b. In the commutative map (Figure 22.1a) node A might refer to a stimulus (S), node C to the associated response (R), and the mediating node B to the complex interior of the organism (O)—where B might be construed as a reflex in Sherrington's sense or a cognitive construct in Tolman's sense. Thus, the $A \rightarrow B \rightarrow C$ path through the digraph of maps provides an elegant representation of Tolman's SOR approach to stimulus-response theory. Skinner believed that this route to explaining behavior was an enormous mistake, and furthermore, was logically unnecessary.

Skinner thought it unwise to attempt to explain observed behavior by postulating unobservable mediating constructs, whose only evidential support arose indirectly from observables. He believed that the observed correlation between stimulus and response is necessary and sufficient for a scientific account of behavior. Skinner justifies omitting node B from adequate explanatory accounts of behavior by arguments borrowed from operationalism, as presented by Bridgman in *The Logic of Modern Physics* (1927). Mathematically, this justification reduces to the claim that behav-
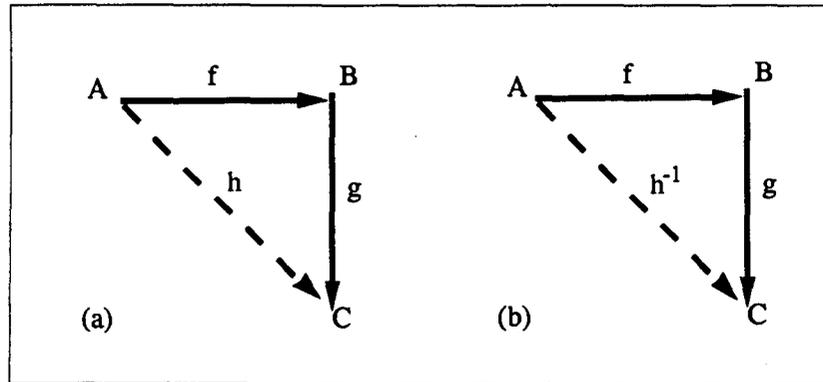
Figure 22.1: (a) Commutative map; and (b) Invertible map.

ioral inverse dynamics satisfies a commutative mapping operation, namely, that a detour path $(A \rightarrow C)$ connecting a set of stimuli, S, with the set of responses, R, can always be found to replace the path mediated by the set of mental operations, $O(A \rightarrow B \rightarrow C)$. That the map is *commutative* enables the operant method to operate only in the exterior frame of reference analogous to mechanics.

Likewise, that the map is *invertible* means that input constraints can act back *onto* the behavioral processes to shape them. Changing the schedule of reinforcement is the central process by which the behavior is shaped in operant conditioning. In Skinnerian terms, if the observed behavior can act back *onto* the schedule of reinforcement that shapes the behavior, then operant conditioning is invertible (i.e., h^{-1} is justified). However, as Skinner correctly assessed, the inverting of Tolman's SOR mapping (i.e., $C \rightarrow B, B \rightarrow A$) is impossible if there are nonlinear constraints introduced in either direction by the complex interior of the organism. For instance, if new meaning can be attached to the S-R coupling by O that is not already present in S, then the mapping would not be invertible (Figure 22.1b). It would become one-to-many from $A \rightarrow B$, many-to-one from $B \rightarrow C$ and, therefore, *into* rather than *onto* from $C \rightarrow B$. This problem inevitably arises whenever there is *context conditioned variability*.[2] For this reason,

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[2] In *context conditioned variability*, a system must adjust its actuators in response to the exigencies of a current context. No general purpose device can carry within its own state description all the parameter values required to make it responsive to the current state of affairs. Hence its behavior is made possible by its lawful descriptions, say, by a system of differential equations, while its behavior on a particular occasion is defined specifically by solutions which satisfy its current *boundary conditions*, that is, by

(a) Skinner's Invertible Map   (a) Tolman's Non- invertible Map

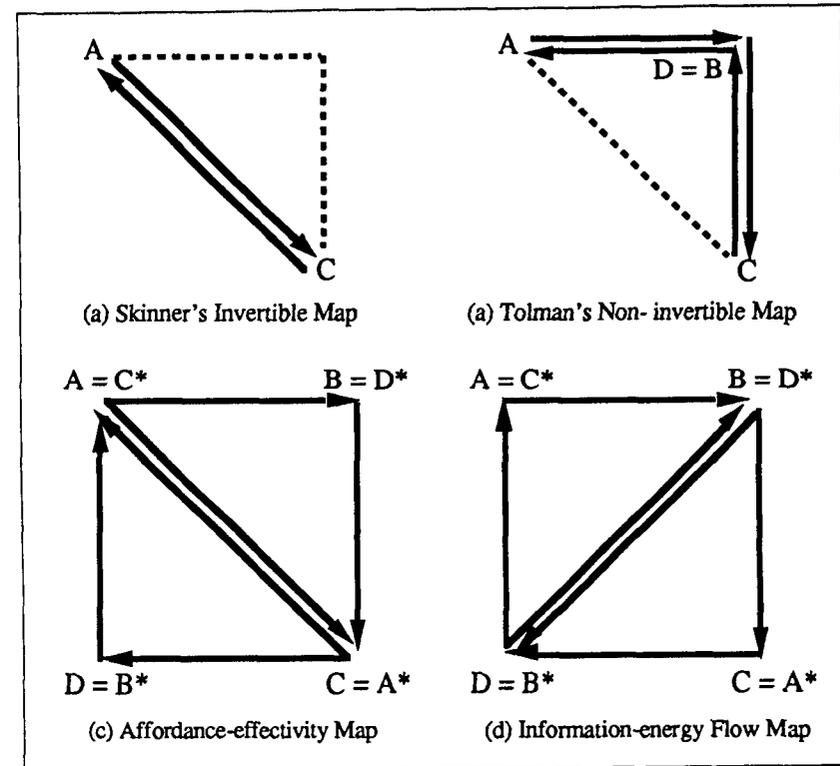(c) Affordance-effectivity Map   (d) Information-energy Flow Map

Figure 22.2: Reciprocities of the perceiving-acting cycle.

Skinner assumed that the simpler behavioral map could be inverted (Figure 22.2a) while the more complex cognitive map could not (Figure 22.2b).

Therefore, by purposefully excluding an interior frame, Skinner endorsed a version of inverse dynamics that remained true to its classical mechanics heritage. Consequently, his psychological rendition can provide no adequate basis for modelling the outflow of intentional acts coordinate with the inflow of information. Specifically, this view of inverse dynamics

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environmental constraints. Consider an organism with the general ability to locomote (its dynamical equation) that must move over a given uneven terrain to a specific target in a particular manner (e.g., carefully but quickly). The exigencies of the particular terrain, its current target, and the manner of approaching that target are the context of variability to which the organism must be locally conditioned. These are the boundary conditions which individuate its general behavior on that particular occasion.

ignores the fact that the significance of the stimulus rests upon its informative value. Hence, it is inappropriate to treat it merely as a punctate application of force that determines the organism's behavior. Treating the stimulus as analogous to an extrinsic force, assumes that intentions are imposed on O from the exterior frame.

In spite of the shortcomings of his approach, Skinner correctly assessed the difficulty encountered by Tolman's view. Inverting the mapping between stimuli shaped by cognitive operations and the resulting behavior requires reciprocity across interior and exterior frames of reference. Unfortunately, the Tolmanian view offers no principled way to achieve this mapping nor its inverse for reasons to be discussed later.

Let us summarize the argument so far. Intention can not be defined simply as an exterior frame reciprocity (i.e., between S and R) because intentions imposed from the outside do not allow for an organism to be self-motivating. On the other hand, if intentions are defined solely from the interior frame (operation on O by O), then they can not reciprocally influence behavior and would have to remain unexpressed. Hence no behaviors of an organism could be influenced by such insular cognitive processes. Clearly, the former deficiency precludes behavioral inverse dynamics from addressing intentional systems; and the latter deficiency precludes cognitive inverse dynamics from doing so. Contemporary representational theories of mind are logically equivalent to this last approach. It is allowed that this view degrades to a solipsistic viewpoint even by its proponents (Fodor, 1981; Fodor & Pylyshyn, 1981), and is accused of being irretrievably so by its critics (Turvey, Shaw, Reed, & Mace, 1981).

One might conclude that because both the behavioral and cognitive approaches to inverse dynamics fail, then this method has no application to psychology at all. We believe this to be an unwarranted conclusion. Perhaps what is wrong with the cognitive approach is not quite what Skinner maintained. Perhaps its map fails the invertibility test, not because it attempts to cut across too many frames of reference, but too few. In the next section, we provide reasons why this might be so.

22.2.3 Inverse dynamics as an ecologically real map

James J. Gibson (1966, 1979), like Tolman, would disagree with Skinner, although for different reasons, that the organism (O) is merely a "throughput system." For Tolman, cognition can embellish the stimulus, while for Gibson, stimulation must be informative about the environment in ways that a stimulus, as a physiological "goad" or a reflexive "force," could never be. They both endow O with a complex interior—which Tolman cites as the residence of cognitive functions and Gibson as the seat of a

tunable (not necessarily linear) information detection operator which resonates to qualitative environmental properties (i.e., affordances). Gibson differs from Tolman, however, in a crucial way which makes the inverting of the cognitive map possible. For Gibson, the environment that surrounds an organism is *real* and objective to each given organism. But the environment studied by the ecological psychologist is neither the environment studied by the traditional psychologist or biologist, nor the physical world studied by the physicist.

Classical physics treats the environment of energy as an exterior frame of reference, ignoring any particular perspectival reference (mapping) it might have to a given organism's internal degrees of freedom. In such an environment, forces are potential energy differences and motions are to be studied as the direct and immediate consequence of the play of forces on matter. If the modern physicist recognizes that objects have interiors, it is just in order to keep the books balanced on the conservations. For often energy, momentum, charge, or some other conserved quantity seeps into the interior frame and must be reckoned with or the books would not balance.

The biologist recognizes the existence of a complex interior field of forces, with mass and energy transport processes, whose consequence may be movement of limbs by transport of neural signals that change muscle tonus. Conversely, the biologist treats the exterior field at a simpler level, recognizing no more complexity than is necessary to keep the books on the distribution of metabolically conserved quantities—whose raw materials enter from the outside and whose waste products are returned there.

The behaviorist psychologist chooses to ignore the existence of the complex interior to organisms for the methodological reasons already outlined, while the cognitive psychologist tends to become lost in its nonlinearities, again for the reasons provided.

None of these approaches recognizes, however, an environment of information which dynamically links a socially invariant exterior with both a biologically invariant interior frame, on the one hand, and with a still more exterior physically invariant frame on the other. That psychological inverse dynamics must couple energy with information across a frame exterior (observable) to one and interior (controllable) to the other, and vice-versa, defines what is meant by an *ecological map*. These reciprocity relations must be carefully extricated from this tangle of variable scales and frames if the exact role of each reciprocity map comprising the perceiving-acting cycle is to be portrayed accurately. This we attempt next.

22.2.4   Extricating the ecological map from the perceiving-acting cycle

A primary behavior of any organism is to orient its biological states to a goal—such as locomoting in a certain manner to a new place (place-finding) or discovering the best path to the goal (way-finding) (Gibson, 1979). What kind of symmetric mapping must the intentional system perform in successful *place-finding* and *way-finding*? Whatever else this mapping might be, it must be performing something analogous to inverse dynamics—but it can not be the classically construed paradigm because, as pointed out, the reciprocal influences cut across interior and exterior frames of reference.

The ecological map must make explicit what Gibson (1979) has called *animal-environment mutuality*, and others have called animal-environment duality, or reciprocity (Shaw, Turvey, & Mace, 1982; Lombardo, 1987). Two fundamental reciprocities compose this more general reciprocity: the reciprocity between the interior and exterior frames of reference and the reciprocity between force and flow. The force-flow reciprocity is between the control of goal-directed activities (generation of forces) and the information detected over the space-time path joining the actor to its goal (generation of flows) (Shaw & Kinsella-Shaw, 1988).

The notion that reciprocities exist such that information may flow across the boundaries of the interior and exterior frames of reference is central. Behavior proceeds following a logic of circular dependence. It is alternately dependent on (1) information inflow, specific to environmental targets, which controls the outflow of intentional acts, and (2) the controlled manner of outflow of behavior, which produces changes in the inflow. This *perceiving-acting cycle* is autocatalytically driven by information, sustained by energy flows, and directed by goals and intentions.

Our aim is to express the perceiving-acting cycle by explicitly seeking a higher-order reciprocity than those used by the behavioral or cognitive approaches. A *higher-order* map coordinates reciprocal relationships over more variables and frames of reference than the behavioral or the cognitive maps. Following Gibson, we seek one that explicitly relates the *interior degrees of freedom* (organismic frame of reference) to the *exterior degrees of freedom* (environmental frame of reference) via information and energy flows specific to a goal selected by an intention. The resulting complex map is our explicit formulation of the organism-environment reciprocity theorem which all variant forms of ecological psychology share. We expect this map will also serve as a cornerstone for ecological physics.

The perceiving-acting cycle continues iteratively from the formulation of intention to its satisfaction when the goal is attained. This iterative cyclicity, reciprocally relating perceiving and acting, constitutes a *complex*
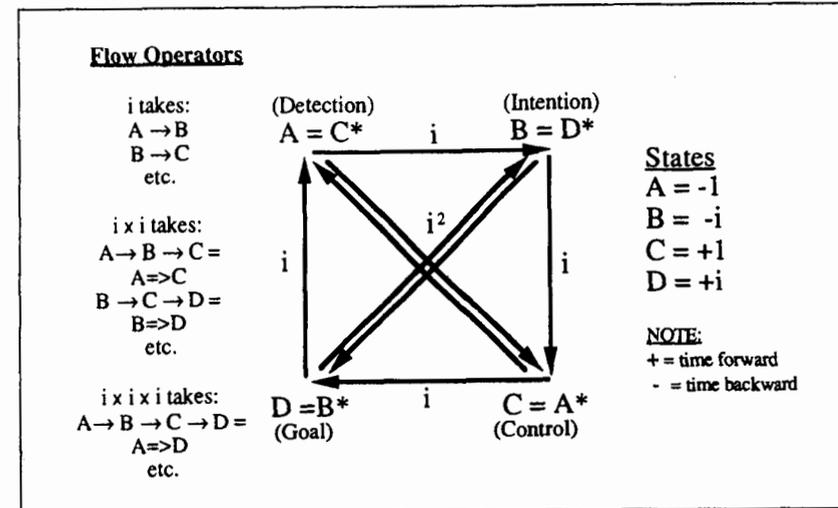
Figure 22.3: Perceiving-acting cycle map.

involutional group. (See Figure 22.3).

22.2.5   Group of complex involutions

The circular logic of causal dependency determines the animal-environment reciprocity across the interior and exterior frames of reference. The ecological map capturing the reciprocities of the perceiving-acting cycle should make explicit the following proposition:

Detection (A), Intention (B), Control (C), and Goal (D) are elements (operations) of a transformation group G, known as the group of complex involutions. These elements form a cyclic group: $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$, under multiplication by a flow operator, $D = i$. The reciprocities of interest, namely, those invertible maps required for inverse dynamics, emerge naturally as a real subgroup g_1 and an imaginary subgroup g_2 of the transformation group (G) under the flow operator.

Let the group of complex involutions be represented by

$$G = \begin{pmatrix} \text{detection:} & A & = & -1 \\ \text{intention:} & B & = & -i \\ \text{control:} & C & = & 1 \\ \text{goal:} & D & = & i \end{pmatrix}$$

Figures 22.2 and 3 highlight the important properties of this group. The

identity element is obviously $C = 1$, since no element operated upon by C changes. And it is easily shown that each element has an inverse.[3] Not only is the group closed, it is cyclic[4] with a generator $D = i$.

Other properties of interest are whether the group $G = (A, B, C, D)$ has any real or imaginary subgroups. Indeed, G has one of each and they are also cyclic. The real subgroup, g_1, is $g_1 = (C, A) = (1, -1)$, where C^n is its generator; namely, $C^n = (-1)^1 = -1$ and $(-1)^2 = +1$, etc. Finding the imaginary subgroup, g_2, requires an assumption about complex numbers. Common practice follows Hamilton's geometric interpretation that a complex number is an ordered pair of real numbers related as follows: where if $1 = 0°$ and $i \times i = -1 = 180°$, then $i = 90°$ and $i \times i \times i = -i = 270°$. Under this interpretation we can multiply any two numbers as follows:

$$(a + bi) \times (c + di) = (a, b) \times (c, d) = (ac - bd, ad + bc)$$

Using Hamilton's formula, the identity is $D = i = 1 \times i = (0, 1)$, and for $g_2 = D^n = i^n$ as the generator, we have

$$D^1 = i^1 = 1 \times i = (0, 1) \qquad \text{and}$$

$$D^2 = i^2 = [0 + (1 \times i)] \times [0 + (1 \times i)] = (0, 1) \times (0, 1) =$$

$$[(0 \times 0) - (1 \times 1), (0 \times 1) + (1, 0)] = \{[0 - (1)], [0, 0]\} = (-1, 0)$$

Therefore, under Hamilton's interpretation of complex numbers as a relationship between real sets of numbers, we have by analogy $g_2 = (B, D) = (-1, 1)$. This analogy can be seen when these subgroups are summarized in the Hamiltonian notation:

$$g_1 = [(1, 0), (-1, 0)] = (C, A), \text{ a real subgroup}$$

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[3] An element g has an inverse if there is some other element g^{-1} such that $g \times g^{-1} = e$, where e is the identity element (i.e., for G, $e = C = 1$). Thus,

$$
\begin{aligned}
A \times A^{-1} &= A \times A = 1 \\
B \times B^{-1} &= B \times D = 1 \\
C \times C^{-1} &= C \times C = 1 \\
D \times D^{-1} &= D \times B = 1
\end{aligned}
$$

[4] A group, G, is said to be *cyclic* if its closure is generated by the powers of a single element. $D = i$ is a generator of the group, as well as $B = -i$. Example:

$$
\begin{aligned}
i &= i^1 \\
i \times i &= i^2 &= -1 \\
i \times (i \times i) &= i^3 &= i \times -1 &= -i \\
i \times (i \times (i \times i)) &= i^4 &= i \times -i &= +1
\end{aligned}
$$

Thus the closure of the group $(i, -1, -i, +1)$ is generated by i^n, where n is an integer. Therefore, the group G is cyclic.

$$g_2 = [(0, -1), (0, 1)] = (B, D), \text{ an "imaginary" subgroup,}$$

such that

$$g_2 = ig_1 = i(-1, 1) = (-i, i)$$

The group G can now be expressed as comprising two real subgroups related by i:

$$G = g_1 \bigcup ig_2 = \{[(1, 0), (-1, 0)], [(0, -1), (0, 1)]\} = [(C, A), (B, D)] \quad (22.1)$$

There can be no more subgroups. For instance, there are no subgroups of order 3, because none of these sets are closed under multiplication.

22.2.6 Relevence of the group G

Before further discussing this group, let us review its relevance. The purpose is to represent the closure of the perceiving-acting cycle under the flow of intentional control, and to express the reciprocities over its variables that naturally arise when the intention is satisfied. Thus this group of mapping operations makes explicit the differences intuitively observed between the behavioral and cognitive maps, and shows their failure to satisfy the logical requirements for a psychologically relevant version of inverse dynamics. It also prescribes properties that a more adequate mapping must satisfy, and justifies our claim that an ecological map exists which satisfies a version of inverse dynamics. Finally, given this abstract specification, we might then attempt to construct the psychologically relevant map.

Our first dividend from applying this group to psychological maps is that it predicts the criticism of Tolman by Skinner. Since the cognitive map is of order 3, then it can not be closed and, therefore, it is not a group. This lack of closure is a common feature of nonlinearity being manifested in Tolman's cognitive map as an addition of elements of meaning by O not to be found in S. Figure 22.2b indicates this possibility by $D = B$. If we make D a separate node in the map, as shown in Figures 22.2c and 22.3, then the additional mappings $C \to D$ and $D \to A$ provide sufficient structure to raise the order to 4 so that closure might be obtained. This was indeed proven to be the case for the group G. Regarding the behavioral map, we see that it does indeed correspond to some group but one that is a subgroup, g_1, of G. If we have made our case that intentional behaviors can not be defined by anything less than the map G, then no proper subgroup could suffice.

The formalizing of these criticisms may seem trivial and unnecessary since words alone may suffice. But defending the use of a method of inverse dynamics in psychology is a formal task and well worth the effort. For the

goals of cognitive and ecological psychology depend on both its validity and our understanding of its proper use. Other dividends will be declared as we untangle the mappings required for systems with intentional dynamics.

22.2.7   Reciprocal maps of the perceiving-acting cycle

Consulting the Maps 22.2c, .2d and 22.3: Let A = detection, B = intention, C = control, and D = goal. Here B represents the formulation of an intention to seek a particular goal D, C represents control the organism must execute given A—the detection of goal-specific information. Adding the goal variable, D, to the psychological maps as an independent node, has a fortunate consequence: It makes inverse dynamics possible! Because the map $B \Rightarrow D$ commutes with the maps $B \rightarrow C \rightarrow D$, and $D \Rightarrow B$ with the maps $D \rightarrow A \rightarrow B$, then we have the invertibility demanded by inverse dynamics. Notice that the map $D \Rightarrow B$ inverts $B \Rightarrow D$ and vice-versa—which is exactly what is needed. Indeed, since this is a group, all of the maps have inverses. For instance, the single map $A \rightarrow B$ is inverted by the commutative triple map $B \rightarrow C \rightarrow D \rightarrow A = B \rightarrow A$. All of the single maps are likewise inverted by the commutativity of the i^3 flow operator, just as all double maps are inverted by the commutativity of the i^2 flow operator. And, clearly, the i^4 flow operator is equivalent to an identity, or no flow operator.

All operators with a positive sign ($+1$ and $+i$) describe flows through the states in a *hereditary*, or (causal) time-forward ($+t$) direction. Likewise, all operators with a minus sign (-1 and $-i$) describe flows in an *anticipatory* or (acausal) *time-backward* ($-t$) direction. The different senses of temporal flow simply means that one direction of flow ($-t$) carries information about the potential future states of the system or tunes the system's controllers in anticipation of future states (e.g., how long to contact with the target), while the other direction of flow either moves the system or "records" for the system (in the sense of remembering or learning) its movements from past states. Thus there is a temporal reciprocity carried by the temporal sign of the maps.

Other important reciprocities should also be noted: The pair of real maps ($+1$ and -1) act on states in the exterior frame while, conversely, the pair of imaginary maps ($+i$ and $-i$) act on states in the interior frame. Hence $A \rightarrow B$ and $C \rightarrow D$ are information and energy maps from exterior states into interior states, respectively, while $B \rightarrow C$ and $D \rightarrow A$ are energy and information maps from interior states to exterior states, respectively. The commutative maps $A \Rightarrow C$ and $C \Rightarrow A$ are clearly inverses, and represent the flow of detected information, on the one hand, and the flow of controlled energy, on the other.

Finally, there is the time-forward energy flow map, $B \rightarrow C$ and $C \rightarrow D$, and the time-backward information flow map, $D \rightarrow A$ and $A \rightarrow B$, that we have already discussed. But given the inverting action of an i^3 flow operator, a reciprocal pair of flow maps may be gotten from the first pair, namely, a time-backward energy flow map, $D \rightarrow C$ and $C \rightarrow B$, and a time-forward information flow map, represented by $A \rightarrow D$ and $D \rightarrow C$. What might these inverted temporal flow maps mean? Operationally, the time-backward energy map specifies that an energy-flow might act as a feedforward, or anticipatory ($-t$) map, as when a cat spring-loads its hindlimbs in anticipation of a jump to a fence top; or a hostess adjusts the set-point on a thermostat in anticipation of keeping her guests comfortable during a dance party. Likewise the causal ($+t$) information flow map, might designate the hereditary effects of past experience on the control of current performance, as when how one performs toward a goal follows from some skill learned or some fact remembered.

So far we have identified three pairs of reciprocities that furnish the essential richness required for a psychological inverse dynamics: reciprocal temporal flows, reciprocal energy-information flows, and reciprocal interior-exterior flows. However, there are still other reciprocities to be considered.

The *affordance-effectivity* reciprocity map is of special importance to ecological psychology. Affordances define the goals that the environment furnishes. These are properties of the environment that causally support goal-directed acts. Operationally defined, they refer to the dual aspects of environmental properties as both causal support for goal-directed actions and as sources of information specifying how such goals might be realized. Hence affordance properties, as goals for an appropriately designed and tuned organism, must be represented by both an inflow of controllable energy (behavior), $C \rightarrow D$, as well as an outflow of detectable information (e.g., optic, haptic, acoustic, olfactory flow pattern), $D \rightarrow A$. An affordance-goal, then, is a double map, $C \rightarrow D \rightarrow A = C \Rightarrow A$. Alone this map refers to a potential goal rather than an actual goal.

To transform a potential affordance into an actualizable goal, the organism must have available for selection a goal-directed function to constrain its control processes in a manner commensurate with the affordance-goal. This affordance-specific control process is termed an *effectivity*. Its existence is equivalent to the condition that the organism be appropriately designed and tuned. This condition logically requires that for any affordance-goal to be actualized by a given organism, then that organism must have a reciprocal effectivity. There must be an inflow of manner-specific information, $A \rightarrow B$, and an outflow of controlled (metabolic) energy, $B \rightarrow C$. An effectivity is also a double map: $(A \rightarrow B \rightarrow C) = (A \Rightarrow C)$. It inverts the affordance map (i.e., $C \Rightarrow A$) just in case the goal, D, is realizable and the

intention, *B*, holds. This reciprocal affordance-effectivity map provides the direction followed by the perceiving-acting cycle.

There is one final reciprocity map, although we have not yet laid the foundation for it, namely, the *social reciprocity map*. This map expresses the fact that goals are socially objective in that they depend on affordance properties that may be shared. A chair may be sit-on-able (afford sitting) for an equivalence class of individuals just as a liquid may be drinkable, a food edible, or a tool wieldable, over socially invariant conditions. Likewise intentions may be socially shared. As humans, we may both intend to sit, drink, eat, or use a tool in an equivalent but not identical fashion. Logically, this means that to the extent that the affordance-effectivity reciprocity maps are the same over different creatures, to that extent the environment is socially invariant and their intentional dynamics may be the same. If this were not so, then no science of psychology would be possible. The affordance-effectivity map is not restricted to individuals but plays an indispensable role in coordinating activities carried out by a collective of individuals (e.g., a crew rowing a boat, musicians playing together in an orchestra, or insects building a nest).

In the next section, we propose the group of reciprocity maps, not as a model of the perceiving-acting cycle but as a means to measure its success or failure in satisfying a system's putative intentions. This group of complex involutions (invertible maps with commutativity and closure) provides, we believe, a plausible account of the socially invariant, intentional dynamics of living systems—but only in so far as they might be expressed *linearly*. The nonlinear aspects of intentional dynamics are discussed in another paper (Kugler, Shaw, & Kinsella-Shaw, in press).

22.3   Linear intentional dynamics

22.3.1   Intuitive description of intentional dynamics

In *Science et Hypothèse*, Poincaré remarks in passing that the *principle of least action* by which one passes from force-based mechanics to a potential (energy)-based mechanics involves an offense to the mind:

> In order to move from one point to another a material molecule which is removed from the action of all [exterior] force, but bound to move on a [potential] surface, will take the geodesic line—that is, the shortest [goal] path.... This molecule appears to know [perceive] the point to which it is wished to bring it [self], appears to know the time it will take to reach there by following this or that path, and then appears to choose the most

> suitable path. In a sense, this statement holds the molecule up to us as a living and free being. (Poincaré, 1902, Chap. 7)

If we include the bracketed editorial comments that we have added, then hardly a better description of a system with intentional dynamics is to be found anywhere.

Our goal in this final section is to remove the offense to the mind and to make plausible the thesis that intention is expressed in the goal-directed behaviors of self-motivated systems as an invariant of motion and the flow of a potential more general than energy but no less conserved.

Consider a sketch of the basic argument behind intentional dynamics: When intention selects an affordance goal with its corresponding effectivity means for achieving it, then the goal sets up an attractor dynamics (target approach) for energy from the system to flow into (a *sink*) and a repellor dynamics (e.g., optic flow) for the target information to flow out of (a *source*). The group of reciprocity maps, representing the perceiving-acting cycle, is successfully goal-directed (i.e., satisfies an intention) when all of its maps commute so that the information and energy flows cancel (invert) each other if and only if the affordance and the effectivity maps do.

We now need to make a plausible case that there is a common potential field in which information sources and energy sinks coexist and interact. However, our goal in this paper is only to begin an explicit description. (See Kugler, Shaw, & Kinsella-Shaw, in press; Shaw & Kinsella-Shaw, 1988; Kugler & Turvey, 1987 for more on these topics.)

For a flow to exist, there must be a force. A force can be defined as the gradient of some potential. (See Kugler & Turvey, 1987, "On Why Things Flow," pp. 64-107). A goal can be said to exert an *attractive force* on the system. We suggest there is some kind of potential difference between the endpoints of a goal-path. For this to be more than mere metaphor, we must find some way of allowing the interior gradient of the organism's metabolic potential to interact with the exterior force field of the environment. This can only take place through the detection of perceptual information which, in turn, must guide the controllers of the neuro-muscular actuators. Hence the relevant potential difference, or *goal-gradient*, can only be defined over an interior (metabolic) potential relating the initial state of intending the goal to the final state of arriving at the goal. This gradient must also reflect the difference between a system's current manner of behavior *where it is* and the desired manner of behavior *where it wants to be*, as defined over the exterior gradients that must be worked with or against in the environment (e.g., gravity, inertia, friction).

The trick is to get the interior gradient and the exterior gradient linearly superposed so that their resultant is the desired goal-gradient. But

this raises another problem. What kind of strange potential is the goal-gradient to be defined over? What is this superposed potential that is neither solely energy nor solely information but both? We call the resulting potential a *generalized action potential*. Whatever this generalized potential is, it is what flows in bi-temporal directions, between the interior and exterior frames, over the perceiving-acting cycle. Furthermore, it is also what must be *conserved* under the intentional dynamics of any system when successfully seeking a goal.

22.3.2 Mathematical strategy

Lie groups were introduced into psychology as a means for identifying the first-order differential equations which embody the information underlying visual integration of patterns. It was argued that a pattern exists as a recognizable entity (i.e., as a Gestalt) if its differential equation remains invariant under a Lie group of transformations (Hoffman, 1968). This claim follows from a classical result by Sophus Lie who showed how Lie groups can be used to identify first-order differential equations they leave invariant. Similarly, we suggest that Lie groups might be associated with first-order differential equations which describe flows as integral curves characteristic of intentional dynamics. In this section we sketch some of the ideas that might help achieve this aim.

For the finite discrete group of reciprocities, $G(A = -1, B = -i, C = 1, D = i)$, to represent the goal-directed dynamical flows of the perceiving-acting cycle requires two extensions: We must find a continuous (Lie) group, G, with which G is homomorphic and we must show that this Lie group remains holonomic (commutative=lawful, see Section 22.3.4) under successful pursuit of a new intention. Being intentionally *satisfied* is identified with the system being in a state of equilibrium so that the resultant goal-gradient is flat. The goal-gradient, defined with respect to goal-specific constraints, is obtained by superposing the gradient of the interior (metabolic) potential with the gradient of the exterior (environmental) potential. A flat goal-gradient means no goal attractor dynamics is imposed on the systems perceiving-acting cycle.

The modelling of the perceiving-acting cycle by a Lie group carries certain stringent mathematical demands which we believe to be offset quite amply by its theoretical advantages. The most stringent mathematical demand is to furnish the Lie group on whose manifold generalized-potential-flows might be defined that are as energy-like as they are information-like. The most certain theoretical advantage is that finding such a *holonomic* group would provide us with generalized motion invariants unique to intentionally motivated flows (e.g., behaviors). The empirical investigation

of the conditions that conserve these invariants should eventually lead us to ecologically lawful descriptions of goal-directed systems instead of concocting arbitrary cognitive rule descriptions of their internalized intentional dynamics. Hence a word about the mathematical strategy.

This superposition of potentials across two different frames will require a *complex extension* of the real Lie groups typically used in classical physics. This *complexification* provides us with the ecological operator that can straddle the interior and exterior fields. We do so by treating the real portion of the complex number field as the basis for one frame and the imaginary portion as another real number field coupled to the first by an operator, i, that is complex (imaginary) but is treated as a real operator.

In addition, we must search for the *compact* simple Lie group that is isomorphic with the complex Lie group initially adopted for modelling the perceiving-acting cycle. This *compactification* process will allow us to take a real simple Lie group associated with a real Hamiltonian flow and extend it to the complex flow of a Hamiltonian over a generalized potential manifold.[5] By compacting the dimensions of a frame (i.e., a field) with sufficient internalized degrees of freedom (at points in the field), we can then accommodate both the exterior and interior flow frames under a more general frame (a superfield). Consequently, information and energy flows, in time-forward and time-backward directions, and within and between frames, all appear under the same *compact real Lie group* as a closed set of automorphisms (flows) onto a single generalized manifold.

At present, we know of no other mathematical approach that is as likely to yield these dividends.

22.3.3 The perceiving-acting cycle as a Lie group

Bear in mind throughout that a Lie group is identified with a topological group which specifies a manifold. More precisely, a Lie group is a continuous group which is also an analytic manifold, and for which the group operations are themselves analytic functions (Gilmore, 1974). (Recall: A function is analytic at a point in Euclidean space if it can be expressed as a convergent Taylor series in some neighborhood of the point.) The analytic

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[5]An ordinary Hamiltonian is a function, H, defined in phase space $(q_1, \ldots, q_n; p_1, \ldots, p_n)$, where the q's are the generalized coordinates and the p's the conjugate momenta, as given by

$$H(q,p) = \sum_{i=1}^{n} \dot{q}(q,p)p_i - L\dot{q}(q,p)$$

structure permits each element of the group to be specified by some coordinate system (not necessarily rectilinear). Moreover, the multiplication of Lie group elements must also be analytic in these coordinates. For instance, if $z = x \times y$, where $\times$ denotes the group operation, then the coordinates of z must be analytic functions of x and y.

The finite discrete group, G, used to represent the perceiving-acting cycle exhibits perfect commutativity, and hence invertibility of its maps. Intuitively, this suggests that no system represented by this perfectly symmetrical group could ever be in disequilibrium. Therefore, it could neither exhibit nor exercise a new intention but must remain in a perpetual state of satisfaction. In order to allow for a more realistic representation, we must make the group more dynamical, that is, we must allow its intentions to unfold over time and space. This can be done rather naturally by letting the group operations be performed in infinitesimal rather than finite perfect steps. By doing so, it becomes possible for the group's flow operations to fall short or overshoot their targets. This makes the original group into a continuous Lie group so long as we restrict the flows to being analytic functions of their coordinates.

A Lie group must be defined on the analytic functions of some coordinates but it is indifferent to the coordinates used. Lie groups are invariant over coordinate frames so long as their degrees of freedom are equal. It does not matter whether the group refers to interior or exterior frames, or time reversed frames, or even to energy as opposed to information flows. All that matters is that the groups or subgroups be commensurate under some coordinate description. This is equivalent to the demand that they commute with some function of the position coordinates—that is, with some kind of a potential which need not, indeed can not be simply (control) energy since (goal) information is an indispensable ingredient as well. If this demand is met, then the goal-directed behavior is identified with a lawfully conserved (holonomic) flow on the manifold of a generalized (informed energy) potential. Our quarry in this hunt is to identify that Lie group which is the required manifold. A Lie group interpretation of a goal-directed activity might be like this.

When a new need or a new desire arises, the intention operator selects the control-parameter values and detects the target-parameter values that define the next desired equilibrium point. The system's next equilibrium point lies somewhere else on the manifold of the generalized potential over which it moves. The location of the system's current equilibrium point is represented by the location of its group's identity element, C. Thus if there is a new intention, B', then for the intention to be satisfied there must be a corresponding displacement of the equilibrium point of the system to a new location, C'. This means that the system is required to "flow"

over a trajectory, ω, to the new target, D', being guided at each step by the detection of the backflow of information, A', from the target, and in a manner that brings its old equilibrium point, C, into coincidence with the new equilibrium point, C', thereby setting $C = C'$. The difference between the generalized potential at the old and new equilibria points specifies the work done on the system by the attractive force of the goal over the goal-path, ω.

This approach encounters at the outset, perhaps, the most serious problem faced by classical (linear) approaches to intentional dynamics—that of dealing with the inherently nonholonomic nature of goal-constraints. For a goal-constraint to be of use, while imposed by the interior frame (by intention), it must prove efficacious in the exterior frame. While agreeing with the intended manner of approach to an intended target, it must also be forceful and capable of doing work in the exterior frame, such as directing the organism's biomass over a distance to the target. Moving over this distance may require work against, in favor of, or indifferent to some exterior gradient.

For this reason, incorporating goal-constraints into the equations of motion poses serious problems for the classical approach to dynamics. For instance, this type of constraint is not simply a function of the coordinates of motion but of a potential difference, a force, as well. By definition, this force is not simply a holonomic constraint (Goldstein, 1980). The effect of this potential difference underlying the force is felt in the exterior frame but has its motivating source in the interior frame.

Hence, unlike all other forces of classical physics, this goal-specific attractive force is not the gradient of a single potential existing in a single frame but is a gradient defined across two different potentials that exist in two different frames. We refer to this as the fundamental problem of nonholonomic goal-constraints.

22.3.4 Constraints: holonomic and nonholonomic

Constraints limit the motion of a system. For an unconstrained system, one merely inserts the initial conditions into the (differential) equations of motion, turns a notational crank, and the predictions automatically grind out. But this is an oversimplification since most natural systems have constraints.

Constraints restrict the physical degrees of freedom of a system in one of two ways: *Holonomic constraints* restrict without requiring material instantiation. Hence they do not materially alter the system. By contrast, *nonholonomic constraints* are able to restrict only because they are physically instantiated—they require some mechanism that materially alters the

system.

Consider the following examples of constraints: the walls of a vessel that constrain the motion of gas molecules, a rail that constrains a speeding train, a path of minimal potential (e.g., with gravity) down which a particle (e.g., water) flows, or the string that restricts a rock swung around the head from flying off on a tangent vector. All of these are examples of holonomic constraints. The term holonomic comes from two Greek roots: Holos—meaning whole, and nomos—meaning law. Physicists have combined these roots to refer to constraints governing any phenomenon that can be expressed wholly as lawful (analytic) functions of their coordinates, and, perhaps, time. A system whose behavior is completely explained by such integrable constraints, is linearly predictable and said to be holonomically constrained. Holonomically constrained systems have one thing in common: They may all be expressed by equations connecting their coordinates of motion.

Nonholonomic constraints are simply those that can not be expressed in this way. More precisely, systems that are constrained, by definition, have some coordinates that are independent. In the case of holonomic constraints, the equations of constraint can be used to eliminate the dependent coordinates (Goldstein, 1980). If they can not be so eliminated, then the constraints are said to be nonholonomic.

A simple example of a system that is nonholonomically constrained is a wheel rolling vertically over a surface without slipping. Two sets of coordinates are required to express this motion: a set of coordinates to describe the location of the wheel's point of contact on the surface, and angular coordinates to describe the orientation of the wheel to the surface. Rolling is the "constraint" that connects these two sets of coordinates and renders them dependent.

The existence of constraints on the motion of a system can not be observed directly. They may only be inferred from its behavior, and from attempts to rewrite its coordinates in some independent manner.

22.3.5   Holonomic and nonholonomic goal constraints

Let us treat an intention, which restricts the behavior of a system, as no less a constraint than the walls of a container, a rail for a train, or a string on a rock.

In the case of intentionally constrained behavior, there are two sets of coordinates as well—those expressing where the system is at the beginning of the movement (initial conditions). and those that express where it must end up if it satisfies the intention (target parameters). The manner (style) of movement over the goal-path connects the two sets of coordinates and

renders them dependent. Does this mean that intention is is necessarily a nonholonomic constraint? Not if this dependency can somehow be eliminated by finding a way to rewrite the equations of motion for the system. The aim of the chapter is to show how it might be done.

If intentions are holonomic constraints, then no mechanism is required beyond that which exploits laws relating energy and information in some specific way. In such a case, one might argue plausibly that, through evolution or learning, organisms come to exploit existing laws very effectively in achieving their goals—without necessary recourse to rules of behavior or "internalized" models of goal-paths, etc. On the other hand, if they are nonholonomic, then since rules require mechanisms to be applied, something like cognitively internalized models of the environment and the actors place in it would have to be assumed.

From a classical physical exterior frame perspective, it appears that a goal (or its intention) can constrain in both ways. It acts *holonomically* whenever the organism acts like an inanimate particle by following the exterior potential gradient (e.g., a rock rolling down a hill and hitting a tree). On the other hand, it acts *nonholonomically* whenever the system acts animately against an exterior potential gradient (e.g., a person running down a hill accelerating by gravity but braking and stopping short of the tree).

Abstractly, maintaining the exterior frame perspective, a system is *holonomically constrained* by a goal over those intervals of the goal-path where the interior potential is inoperative or in stationary process. These will be those integrable (open) intervals of the goal-path between choice-points. A system is *nonholonomically constrained* when its behavior must be controlled across choice-points. These will be those (closed) intervals that include choice points. Choice-points act as nonintegrable constraints and denote regions in the exterior frame where the the goal-path curve becomes nonanalytic (discontinuous). How might this happen?

A system with an active nonstationary interior gradient can be thought of, metaphorically, as depositing singular points along its trajectory in the exterior frame where choices may be made (e.g., to brake, change direction, speed-up or slow-down). These control decisions arise at those points along a trajectory where the system must inject a sustaining squirt of interior field potential to keep moving in the same manner toward the same target, or where it can counter the work done on it by an exterior gradient. Therefore, we call these *choice-points*.

Mathematically, the existence of choice-points represents regions in the exterior frame where the goal-path geometry becomes *compact*, in the sense of hiding additional (interior) degrees of freedom at singular points along the path. (In physics this is sometimes called the Kaluza-Klein strategy).

Hence because these compact singularities determine the point of contact between the two frames, there is no way to integrate the two potentials, and thereby treat them as equations of constraint. Normally, these equations would simply be added to the equations of motion (by elimination) and the resulting system of differential equations solved (integrated) to determine the system's path of motion. Unfortunately, where goal-points crop up, this cannot be done.

From the exterior frame perspective, the problem is exacerbated by the fact that goal constraints must satisfy the final conditions of the involved differential equations as well as their initial conditions. Thus the value of a goal constraint can not be found until after the equations of motion for the system are solved (that is, until after the system reaches its goal), yet its value is needed to evaluate the integrating factor before the motion equations can be solved (integrated). Hence a vicious cycle! The problem is compounded when intentions are not stable because this process must be carried out as piecewise integration between each pair of choice-points. The final yield will be a goal-path in the exterior frame that is a mixture of concatenated holonomic and nonholonomic subintervals.

If, however, we can provide a theory relating the interior frame to the exterior frame, and vice-versa, then there is the possibility that constraints that are nonholonomic in one frame might prove holonomic over both frames. This is equivalent to claiming that there is an invariant of motion, a conservation, that carries the system holonomically over these singular regions without a cognitive mechanism mediating the behavior. What might qualify as such a motion invariant?

22.3.6   Intention as a "holonomizing" constraint

Nonholonomic goal and choice-point constraints are "exceptional boundary conditions" (Kugler & Turvey, 1987, p. 405) on the equations of motion. Elsewhere, we have identified the source of the exceptional constraints with intention as an operator informed ahead of time by goal-specific information (e.g., you see where you are going before you get there) (Shaw & Kinsella-Shaw, 1988). To reiterate: Such exceptional boundary conditions would enter into ordinary equations of motion (say, Lagrange's) as nonholonomic (nonintegrable) constraints that cannot be eliminated in the usual way (say, by Lagrange multiplier method).

In a nutshell, the fundamental problem for a classical mechanics approach to intentional dynamics is to show how goal-constraints can behave holonomically at some higher-order, proper scale of analysis. With the concept of the *real ecological map*, we have nominated the ecological scale as the proper scale of analysis and identified holonomy with the group which

satisfies the *organism-environment reciprocity* espoused by Gibson (1979).

22.3.7   Ecological work and goal-attractive forces

The work produced by the force of a goal-constraint acting over a goal-path, ω, can be represented by

$$\int_C^{C'} F \, d\omega = \tilde{V}(C') - \tilde{V}(C) \tag{22.2}$$

where $\tilde{V}$ is the generalized potential at each endpoint.

Moreover, this work integral corresponds to the relative steepness of the goal-gradient over the goal-path from the initiation of intent (goal-selection) to its satisfaction (reaching the goal). The work required to move the biomass of the system *toward* the goal is exactly the same as the work required to produce a backflow of information from the goal. We do not, therefore, have any basis for distinguishing the work requirements for the information backflow from those of the biomass forward flow. This conjugacy of work in the time-forward direction with the information flow in the time-backward direction is illustrated in those cases where the anticipatory adjustments of a system to an expected future state are directly proportional to the influx of information.

For example, a person standing upright in Lee's (Lishman & Lee, 1975) gliding room will sway in the direction that the room sways. The magnitude of these coordinate oscillations are equal. Hence the radial flow of optical texture from the frontal wall of the movable room specifies to the person that he or she is posturally unstable when in fact it is the room that is unstable. Consequently, the subject imposes muscular impulse forces to counter the onset of the expected disequilibrium. In this way the causal energy flow is conjugate to the acausal information flow because both are dominated by an intentional flow.

A similar case is where subjects in a flight simulator, instructed to maintain constant altitude, nevertheless change altitude as a function of the change in the edge rate of optical ground texture flow (Rik Warren, personal communication, 1988). Again we see that the perceiving-acting cycle is organized by an intentional flow into reciprocities which impose a symmetry (conjugacy) on the lower-order exterior (target) information flow and interior controlled (metabolic) energy flows. There is a similar breakdown in holonomy of information and control.

These are paradigmatic cases that illustrate the reciprocal dependencies of information and energy flows: The intentional flow is an attempt to displace the equilibrium point of the system to a new location on the

generalized potential manifold so that the old identity element at C is made coincidental with the new one expected to be at C'. Of course, since the person chooses the wrong bifurcation at the choice-point, a nonholonomic perturbation is introduced into the system. If the person had chosen correctly, then posture (e.g., uprightness, altitude) would have remained stationary. Successful goal-directed actions require that the perceiving-acting cycle preserve holonomy over its conjugate information and energy flows. Intention plays the role of a *generating function*[6] which couples information and energy across the interior and exterior frames. (Exactly how this is done is discussed in detail later.)

Many additional examples can be given that exemplify how realistic intentions impose holonomic organization on the perceiving-acting cycle (e.g., running to catch a baseball, braking a car when approaching a crosswalk, etc.). Again our aim will be to sketch mathematically the (linear) requirements for a system with successful (holonomic) intentional dynamics.

From the above examples, we have evidence that the repellor (source) dynamics of the goal and its attractor (sink) dynamics are dual-counter flows around a closed circuit, as indicated by $B \rightarrow C \rightarrow D$ versus $D \rightarrow A \rightarrow B$ mappings in G. This means that the following must also be true

$$\oint_\omega F \, d\omega = \left(\tilde{V}(C') - \tilde{V}(C) \right) - \left(\tilde{V}(C) - \tilde{V}(C') \right) = 0 \qquad (22.3)$$

whenever a system moves successfully over a goal-path. Since cyclic coordinates (e.g., action-angles) have a privileged role to play in the description of conserved quantities, this description is propitious.

For a system to be goal-directed means that on the manifold of its generalized potential a goal-gradient exists which constrains the system's motion to the "shortest" (stationary) path. This goal path is a geodesic whose parameters determine the attractor dynamics governing the system's movements. One strategy for making this intuitive description of the intentional dynamics of a perceiving-acting-intending system formally explicit is to represent it by a Lie group, or more explicitly, by the Lie algebra of a Lie group (Belinfante & Kalman, 1972). By showing that there is a valid Lie theory representation of the reciprocity mappings underlying a

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[6] A *generating function* F acts as a bridge between two frames. F is expressed partly in terms of the canonical variables of one frame and partly in terms of the canonical variables of the other frame. For instance, an equation of transformation specified by a generating function $F = F_1(q, Q, t)$ or $F = F_2(q, P, t)$ gives the transformation as a function of both old and new coordinates. For our purposes, we need to use generating functions that bridge between the generalized canonical variables of interior and exterior frames that flow in $+t$ or $-t$ directions: For instance, $F = F_1(Q, Q^*, t)$ or $F = F_2(Q, P^*, -t)$. (For discussion, see Goldstein, 1980, pp. 378-390).

goal-directed perceiving-acting cycle, we would be guaranteed of the commutativity (holonomy) of all of its reciprocity maps. This result would, therefore, give assurance that a method of inverse dynamics might, in principle, be developed for psychology.

22.3.8   Holonomy as Lie algebra bracket products

Let us keep in mind our paradigmatic cases (the gliding room and flight simulator examples) for illustrating the breakdown of holonomy of the perceiving-acting cycle. How might we express the degree of holonomy of a goal-directed system? Here is one way.

A Lie algebra adds a new operation to the Lie group called the *bracket product*. Its bracket operator is represented by $[u, v]$ and is defined by

$$[u, v] = (uv) - (vu) \qquad (22.4)$$

This idea can be generalized to a pair of vector fields X, Y on a manifold M, their (Lie) bracket $[X, Y]$ is the vector field whose value is the difference between the vector products $X Y$ and $Y X$. The bracket operator is said to be a commutator on the vector space when $[X, Y]$ shows no linear deficiency, that is, when

$$[X, Y] = (xy) - (yx) = k, \text{ for } k = 0 \qquad (22.5)$$

for all $x \in X$ and all $y \in Y$. Intuitively, the bracket operator, when it acts as a perfect commutator, guarantees the invertibility of a given mapping between the two fields. Here X might represent the field at C and Y the field at C'; then the product $[X, Y] = Z$ represents the flow over the goal-path from C to C'. If $Z = k$, for $k \neq 0$, then k is a measure of the breakdown of holonomy.

Thus, in addition to the commutators defined on the powers of the involutional flow operators, i^n, we now have a bracket operator. The bracket product is not only bilinear but anticommutative as well.

$$[x, y] = -[y, x] \qquad (22.6)$$

And finally we require that the bracket operator satisfy the Jacobi identity:

$$[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0 \qquad (22.7)$$

Lie algebras commute only if their associated groups do. But Lie groups sometimes have successive transformations that do not commute, or, equivalently, that exhibit a lack of linear lawfulness. When they do, then they are said to be *nonholonomic* (Nelson, 1967). Holonomy is the general idea that describes what happens when we can perform a cycle of transformations,
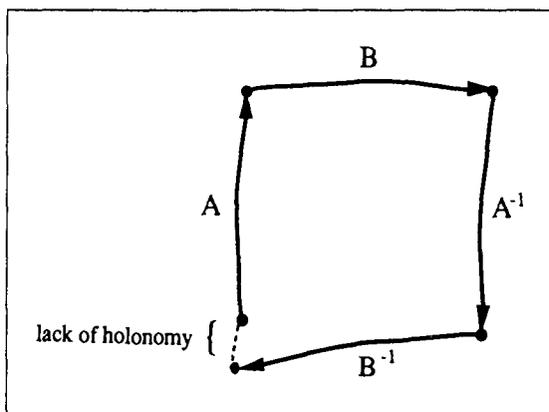
Figure 22.4: Lack of holonomy (commutativity) in a Transformational (flow) cycle, $ABA^{-1}B^{-1} = ABCD$, of G_L.

$ABA^{-1}B^{-1}$, which retains closure as opposed to failing to close. Figure 22.4 shows a breakdown in the holonomy of the bracket operator

$$[A, B] = AB - B^{-1}A^{-1} = ABCD \neq 0 \qquad (22.8)$$

It is important to distinguish causal (energy) flows from acausal (information) flows, and these from intentional flows. For although these flows may be supported by the same flow mechanism—the physical instantiation of the perceiving-acting cycle—they may have different degrees of holonomy. That is, some flows may be holonomic across all maps while others may breakdown for some maps. There are several ways in which these flows that share the same causal cycle may differ in degree of holonomy.

As pointed out earlier, information and the energy flows are causally coupled: When an animal moves through the environment a flow of information covaries with the flow of energy. For instance, moving toward, away from, or lateral to a target will be accompanied by and, indeed, cause a radial outflow, a radial inflow, and a lateral flow of associated information. If the information is specific to and satisfies the stipulated goal-constraints, then the system is behaving toward the target as intended. Therefore, all of the flows are mutually holonomic.

On the other hand, if the information flow does not satisfy the goal-constraints, then the flow of information is, of course, no less causally holonomic. Here either the energy flow or the information flow or both may be intentionally nonholonomic. Clearly, then, intentional holonomy is a higher-order group property than either causal (energy) holonomy or

acausal (informational) holonomy. Furthermore, as intended acts become complicated, having nested subgoals, then the relative degrees of holonomy of the nested flows similarly become nested.

In summary, we might conclude that the breakdown of holonomy means that the Lie version of the perceiving-acting cycle group, G_L, is no longer perfectly symmetrical in its mappings. Another way in which intentional holonomy may breakdown is when the system switches from one intention (goal-path ω) to another (goal-path ω') and becomes captured by a new goal-gradient. This may happen because, with respect to the old intention, the system lacks adequate control, or sufficient information, or has formulated an unrealistic intention. This new attraction compels the system to make a choice regarding which of the set of goal-paths defined at the choice-point is most likely to re-establish its equilibrium. Finally, after a proper choice, goal satisfaction is achieved by the flow of control (relocating its identity element, from $C \rightarrow C'$) over the new goal-path to some other neighborhood on the generalized potential manifold where the flow gradient is minimized.

22.3.9   Proving new invariants of motion

Intention is the operator that makes goal selections. Both the target and the manner (mode) of approach to the target are necessarily part of this selection. Thus intention sets up the algebraic structure of the perceiving-acting group as a flow of all four group elements over the manifold, $G_L(A, B, C, D) \Rightarrow G_L(A', B', C', D')$. Consequently, if the algebraic structure is to be preserved over the goal-path, then the intention (goal state) must be invariant over initial and final conditions. This can only be so, if intention (goal state) and the other dynamical variables it selects commute with the generalized Hamiltonian[7] and thus with each other.

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[7] A *generalized Hamiltonian*, G, is defined in a generalized (complexified and compactified) phase space $\{(Q_1, \ldots, Q_n; P_1, \ldots, P_n), i(Q_1^*, \ldots, Q_n^*; P_1^*, \ldots, P_n^*)\}$, where Q and P are the generalized (target-specific information) coordinates and generalized momenta (manner-specific movements), respectively, and both are observable in the exterior frame. In the interior frame, accessed by the complex operator, we have dually Q^* and P^*—the generalized (manner-specific information) coordinates and generalized momenta (metabolically produced control impulses), respectively. The generalized Hamiltonian, G, is defined (complexly) adjointly over the exterior, and interior phase spaces by

$$G(Q, P; Q^*, P^*) = \left\{ \sum_{i=1}^{n} \dot{Q}(Q, P)P_i - L(\dot{Q}(Q, P)) \right\} + \left\{ \sum_{i=1}^{n} \dot{Q}^*(Q^*, P^*)P_i^* - L(\dot{Q}^*(Q^*, P^*)) \right\}$$

In particular, preservation of the intention (goal state) over the entire course requires that the four operators of the perceiving-acting group must perform their duties invariantly. The system must continue (i) to detect target information vigilantly lest it become blind and go astray, and (ii) to control diligently its style of approach (mode) to the target lest it behave erratically and go astray. Likewise (iii) the intended goal must causally persist lest the system become aimless. Finally, (iv) the intention must remain globally specific to the goal-parameters lest the system lose its motivation to act toward the goal appropriately or at all. These four operators are dynamical variables because they are functions of the change of the generalized coordinates over time. '

Logically, the first three dynamical variables (operators) are bound together by the action of the fourth one into a coherent goal-seeking entity. Consequently, if the action of the intention is conserved, and no causal thwarts impede the total group action, then so are they all conserved— provided, of course, that they all commute with the generalized Hamiltonian. Three small nonrigorous "proofs" are given in the appendix to summarize the main conclusions that can be drawn regarding this (linear) approach to intentional dynamics.

We begin by showing that the dynamical variables A, B, C, D, all elements of G_L, do indeed commute with one another. We conclude that because all the variables commute with intention, then if it commutes with the total action, G_L, so must they all and thereby be conserved (First Proof). Thus we show next that intention is conserved because it commutes with the generalized Hamiltonian of interest—total action (Second Proof). Finally, we show that intention plays a role in intentional dynamics that is unique. It binds all dynamical variables to the stationary goal-path of a conserved quantity—total action (Third Proof). A lemma on the way to the first proof is introduced to resolve the problem that while G_L appears to require a complex Lie group representation,[8] classical Hamiltonians ap-

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Just as an ordinary $H(q, p, t)$, over a time invariant path in phase space conserves the time-forward flow of total energy (kinetic + potential energy), so the generalized Hamiltonian, $G(Q, P; Q^*, P^*; +t, -t)$ conserves the relationship between time-forward and time-backward flows of energy and information that comprise total (goal-specific) action. Total action, G, is the sum of total control information, $(Q, Q^*, +t, -t)$ and total useful energy $(P, P^*, +t, -t)$ over both frames (i.e., the interior and exterior phase spaces). The result of this superposition of the two potentials (superposition of *passive* transformations) yields a generalized potential difference (a flow) over a trajectory in generalized phase space, the goal-path, ω (treated as if it were produced by *active* transformation, or flow). The gradient of this potential difference is the attractive force of the goal exerted on the system that intends the goal as it traverses the goal-path. (For details see Kugler, Shaw, & Kinsella-Shaw, in press).

[8] We have used the term *representation* in the nonmathematical sense, which is the exact opposite from its usage in mathematics and mathematical physics. A Lie group

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ply only to real flows. (*Please note:* Hereafter we will assume $G = G_L$ except where otherwise noted.)

22.3.10   Summary of results from the appendix

We show in the appendix that if the extended bracket operations hold for the real restriction of the complex Lie group, G^R, so that all of the dynamical variables of G commute (Proof 1), and if these variables all commute with the generalized Hamiltonian (Proof 2), then the system's intentional dynamics are conservative. We also show that this is so because intention plays a unique role as the generator of the total action flow (the Lie algebra) of the perceiving-acting cycle over a stipulated goal-path. Let us refer to this as the *fundamental theorem of an ecological approach to intentional dynamics.* Moreover, applying the Jacobi identity iteratively can, in principle, allow us to uncover a complete sequence of motion invariants. Although in practice this approach is often disappointing—terminating in zeros or constants before new invariants of interest are uncovered—nevertheless it deserves to be added to ones arsenal of potentially fruitful methods.

We have postponed to the last any attempt to formally characterize the generalized action potential, G, because the preceding mathematical developments were needed to do so. We turn to this task now. (But see Kugler, Shaw, & Kinsella-Shaw, in press, for a more extended treatment).

22.3.11   A generalized action potential for intentional systems

The rate of change of the generalized kinematic (total control information) and kinetic (total useful energy) co-parameters provide a complete specification of a goal path, ω, over which total action flows geodetically if an intentional action is to be successful. The course of values of these functions is mutually directed by the "attractive force" (the gradient) of the goal. This means that G also changes as a function of both target information and the intended manner of approach to the target. To accomplish a particular manner (mode) of approach (e.g., locomotive style), the system's interior gradient (metabolic energy flow) must do work with, against, or indifferent to the exterior gradient of the environmental frame (e.g., gravity,

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or a Lie algebra is said to be represented through a homomorphism by a similar object composed of linear operators which in turn act on a vector space. Here the vector space is spoken of as a module over the group or over the algebra. In psychology we would more naturally refer to the module as the object to be represented by the group or algebra. Notice, however, that because the mapping from object to module is homomorphic rather than isomorphic this makes the latter intuitive usage rather chancy.
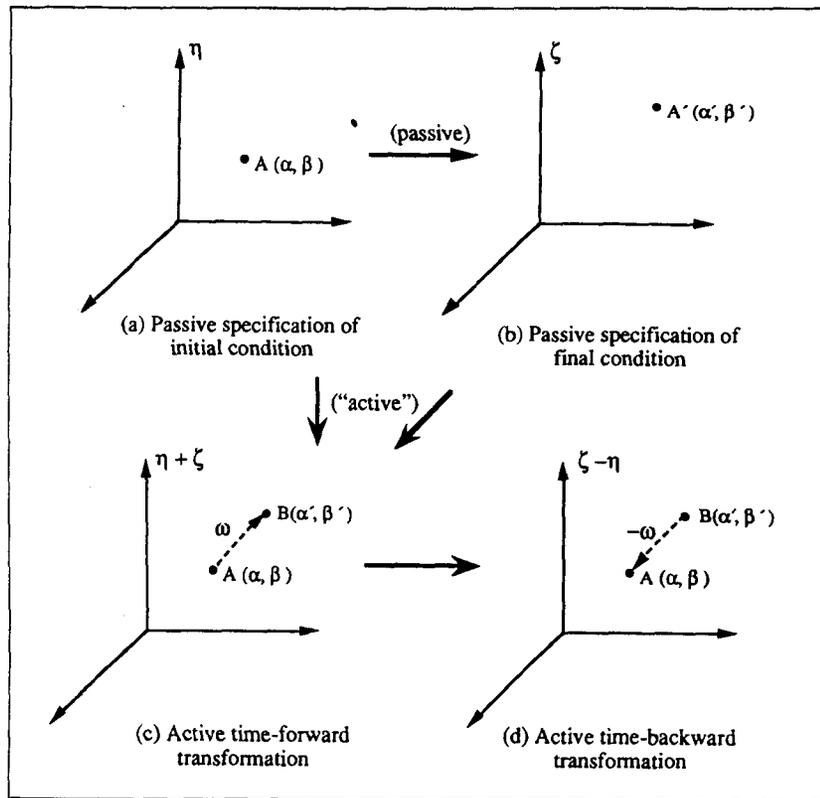
Figure 22.5: Composing an "active" flow transformation from two passive flow transformations.

masses). The intended goal path's gradient, therefore, must result from a superposition of the interior and exterior gradients. This superposition can not be carried out directly because the goal acts as a nonholonomic (nonintegrable) constraint in the exterior frame. But it can be carried out indirectly across frames using a method of coordinate transformation. Consider Figure 22.5.

Coordinates of an exterior phase space may be used to express a goal-path, ω, by plotting a pair of points related under a passive transformation into a new generalized exterior phase space as an active transformation. The Hamiltonian for phase space η expresses the initial conditions at point $A(\alpha, \beta)$ for a system embarking on a goal-directed path (Figure 22.5a).

The Hamiltonian for phase space ζ expresses the reinitialized conditions of the system at $A'(\alpha', \beta')$ some infinitesimal distance further along ω (Figure 22.5b). After a certain amount of time, the system reaches the end of the goal-path—which, for convenience, might also be represented by $A'(\alpha', \beta')$.

No change is captured by the passive coordinate transformation, $\eta \rightarrow \zeta$, (Figures 22.5a and 5b) because a canonical transformation in itself, being symmetrical over the flowpath, cannot "move" a system within a single phase space. We can only express a difference across spaces, represented by the $+t$ mapping, $\eta \rightarrow (\zeta - \eta)$ (Figure 22.5c), on the one hand, and $\zeta \rightarrow (\eta - \zeta)$ (Figure 22.5d), on the other.

This method is indifferent to the frame in which the phase spaces are defined. It can just as readily be applied to pairs of points related transformationally across interior frames, as well as to pairs of points related across interior and exterior frames. Similarly, as shown, this method is also indifferent to the temporal direction of the flow and to what flows—information or energy.

Since this method of composing "active" transformations from passive transformations is the key to a full appreciation of the concept of a generalized Hamiltonian flow as applied to intentional dynamics, let us try to be more explicit.

22.3.12   Total action as a conserved quantity

We have spoken of a system's movement in terms of a single representative point—its equilibrium point. In fact, a moving system is a large swarm of biomass points whose velocities and positions can not be perfectly described by classical mechanics. Let us call the system's points a *statistical ensemble*. Whatever motions this ensemble undergoes, is a function of the myriad initial conditions of all the points in the ensemble. Hence, for the sake of practical description, we adopt a statistical description of these initial conditions. We assume that the swarm of biomass points is a fluid-like substance that flows through phase space. Imagine the fluid moves one infinitesimal step (an infinitesimal contact transformation) for each cycling through the group G. This phase fluid, then, represents the total action of the Lie group on the system, while its stepwise flow represents the iterative application of the extended bracket product of its dynamical variables which vanishes just in case the total action is conserved.

Let D refer to the density of the total action fluid. The time variation of D can arise in two ways: Since the density in the neighborhood of a given point depends implicitly on the coordinates of the target, $D(Q, Q^*, P, P^*)$, as the coordinates change, the system may wander through phase space in pursuit of the target. There may also be an explicit dependence on time,

$D(+t, -t)$ in the sense that the system may expend action potential in order to remain stationary in space. Hence the total time derivative of D is defined over both kinds of temporal variation. This is shown by

$$dD/dt = [D, G] + \partial D/\partial t \qquad (22.9)$$

The Poisson bracket expresses the implicit dependence on time, and the ratio of partial derivatives the explicit dependence. The total derivative is sometimes called the *hydrodynamic* derivative because one follows the density of the fluid over the trajectories. In field theory this is called the *Lagrangian perspective* because individual particle trajectories must be tracked over time. We liken this to an *active coordinate transformation* within the same frame. As argued earlier, this active approach applies only to flows in a single frame. Unfortunately, we need an approach that applies to flows between frames. Luckily, there is another perspective that might be taken on flow, called the *Eulerian perspective.*

This perspective examines flow over different spatial frames while frozen in time in order to reveal any higher-order stationary "currents" that may exist. This is more like a *passive coordinate transformation* where the description over two coordinate frames are compared (as in Figures 22.5c and 5d). When these coordinate frames are treated over time, if the flow is conserved, then the Lagrangian active description and the Eulerian passive description are formally equivalent. Thus the total derivative disappears and we arrive at

$$[D, G] = -\partial D/\partial t \qquad (22.10)$$

where the bracket product is equivalent to an active transformation although it is really a passive transformational description of flow. This proves Liouville's famous theorem that the density of phase fluid, under an infinitesimal contact transformation, is a motion invariant since it remains constant over time. The system as a fluid ensemble is in statistical equilibrium if the density of its fluid is a motion invariant. Since the partial derivative with respect to time expresses any change D might undergo, then this term must vanish whenever the system is in equilibrium. Hence

$$[D, G] = 0 \qquad (22.11)$$

Instead of statistical equilibrium, our terms, of course, have all along been designed to express the condition for intentional equilibrium. Thus this equation asserts a condition that is equivalent to G being perfectly commutative under all bracket products of its dynamical variables. This is exactly the condition that we set out to prove.

22.4 Conclusion

There is good reason to believe that a method of inverse dynamics might be found that will work for psychology. This method however cannot be that of the behaviorists or the cognitivists but requires an ecological real map that satisfies Gibson's reciprocity theorem. Knowledge that the perceiving-acting cycle can be consistently represented by a Lie algebra, whose dynamical variables commute with a generalized Hamiltonian, moves us one step closer to the realization that natural systems with intentional dynamics (or even artifactual ones—robots) may be law-governed rather than rule-governed. Under this view, intention is both naturalized and holonomized.

Acknowledgments

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22.6 Appendix: some proofs

22.6.1 Lemma: The real restriction on a complex Lie algebra

Real Lie groups and their associated *real* algebras are not adequate for dealing with flows that must transact business between the interior and the exterior frame. Recall that to represent the perceiving-acting cycle we need a complex group with a pair of subgroups—one real and one "imaginary"—related by the complex number i. Does this mean that instead of a real Lie group, G, all along, we should have been talking about a complex extension of the Lie group, G^C, as the best means to represent total action of a successful goal-seeking perceiving-acting cycle. This is only partly true, as the following argument shows.

Consider a complex Lie algebra L. A real form L_0 of L can be a subalgebra, called the *real restriction* L^R of L. L^R is then a Lie algebra over the real subfield of the complex number field. Furthermore, if $e_1, \ldots, e_n$ are a basis over the real numbers for the real form L_0, then these same vectors are

a basis for L over the complex numbers. The vectors $e_1, \ldots, e_n; i(e_1, \ldots, e_n)$ are linearly independent over the real numbers, and are bases for the real Lie algebra, L^R. Although i is a complex number, it can serve as a real linear operator on L^R having the properties $i^2 = -1$ and $[x, iy] = i[x, y]$ for all $x, y \in L^R$. Moreover, any vector $z \in L^R$ may be uniquely written $z = x + iy$, where $x, y \in L_0$. Our flow operator can be defined as an automorphism S of the real Lie algebra L^R by $S(x + iy,) = x - iy$, for all $x, y \in L_0$. Hence S acts as a *complex conjugate* operator.

Mathematically, this is what is meant by a reciprocity because it inverts the direction of a vector joining two dynamic variables. This operation is often called an *involutive automorphism*, that is, an automorphism whose square is the identity operator, $S^2 = 1$. S is the identity when restricted to L_0 but its negative, -1, when restricted to the subfield iL_0. This is an important point which we shall use in the second proof below.

By this technique it is possible to have your cake and eat it. We have the complex Lie group needed to represent the complex flows of the perceiving-acting cycle but we have it in the form of a real restriction on the complex number field. In a sense, this is the best of two worlds—a real operator coupling two real flow fields but which performs the duties of a complex operator—a way to straddle the interior and exterior frames without reducing one to the other.

22.6.2 First proof: dynamical variables of G commute with one another

From the definition of the complexified Poisson bracket and by using Jacobi's identity, the bracket product can be redefined dynamically so that

$$\frac{d[\alpha, \beta]}{dt} = \left[\frac{d\alpha}{dt}, \beta\right] + \left[\alpha, \frac{d\beta}{dt}\right] \tag{22.12}$$

is also true. This means that if two (real) dynamical variables of G, α and β, are each conserved, that is, $\partial\alpha/\partial t = 0$ or $\partial\beta/\partial t = 0$, then not only is their (real) bracket product, $[\alpha, \beta]$ conserved but so are all of its (real) linear combinations. For example, here the dynamical variables of interest might be the total useful energy and the total control information (as expressed by the self-adjoint temporal integrals) and the bracket product might represent the total action flow, as discussed earlier. These products and linear combinations of products form a subalgebra of the Lie algebra of all conserved dynamical variables.

Before showing this, however, we need to *complexify* the Lie algebra associated with the group $G(A, B, C, D)$. This can be done by using Leibniz's rule to complexify its cyclic group product: $[ig, G] = i[g, G] + g[i, G]$

for $g \to (A, B, C, D)$ and where i is the flow operator. We recognize $[g, G]$, as the flow (derivative of g) in the direction of the field idG on the goal-path manifold of the total action flow. Here there is no danger of confusion because G is the manifold as well as the Lie group $G(A, B, C, D)$. Unfortunately, there is no way to shorten the algebra required to carry out this complexification procedure. Use of the following identities, however, will help:

$$i[g_1, g_2] = [g_1, ig_2] = [ig_1, g_2] - g_1[i, g_2]$$

where $ig_1 \to g_2$ is the cyclic permutation from one element to another in $G(A, B, C, D)$ under the action of the group operation (i.e., flow operator). Recall that $iA = i(-1) = -i = B, iB = i(-i) = -(-1) = 1 = C, \ldots$ and so forth. The abbreviated real fourfold bracket product for G can be represented by $[[[A, B], C], D]$ and its abbreviated complexified bracket by $i[[[A, B], C], D]$. The full expression of the latter is

$$[[[A, B], C], iD] + [[[B, A], D], iC] + [[[C, D], A], iB] + [[[D, C], B], iA]$$

Next, we evaluate separately the four terms in order[9] and then recombine them to show that

$$i[[[A, B], C], D] = 0$$

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[9] The first term is evaluated thusly

$$[[[A, B], C], iD] + \ldots = [[[A, B], C], A] + \ldots =$$
$$(((AB - BA)C - C(AB - BA))A - A((AB - BA)C - C(AB - BA))) + \ldots =$$
$$(-((AB - BA) - (AB - BA)) + ((AB - BA) - (AB - BA))) + \ldots =$$
$$(-((i - i) - (i - i)) + ((i - i) - (i - i))) + \ldots =$$
$$(((i - i) - (i - i)) - ((i - i) - (i - i))) + \ldots = 0 + \ldots$$

The second term in the bracket product is evaluated similarly,

$$0 + [[[B, A], D], iC] + \ldots = 0 + [[[B, A], D], D] + \ldots =$$
$$0 + (((AB - BA)D - D(AB - BA))D - D((AB - BA)D - D(AB - BA))) + \ldots =$$
$$(((0)i - i(0))i + i((0)i - i(0))) + \ldots = 0 + 0 + \ldots$$

and the third term,

$$0 + 0 + [[[C, D], A], iB] + \ldots =$$
$$0 + 0 + [[[C, D], A], C] + \ldots =$$
$$0 + 0 + (((D - D)A - A(D - D))C - C((D - D)A - A(D - D))) + \ldots = 0 + 0 + 0 + \ldots$$

And, finally, for the fourth term we have

$$0 + 0 + 0 + [[[D, C], B], iA] =$$
$$0 + 0 + 0 + [[[D, C], B], B] =$$
$$0 + 0 + 0 + (((DC - CD)B - B(DC - CD))B) - (B(DC - CD)B - B(DC - CD))) =$$
$$0 + 0 + 0 + 0$$

This means that under the Lie algebra associated with the Lie group with a real restriction, G^R, all the dynamical variables in the group commute with each other because their extended bracket product vanishes. These variables are, of course, flows on the total action manifold associated with the perceiving-acting cycle. This association is unique and specific[10]. This suggests that to the question: "Can the perceiving-acting group remain holonomic as it traverses a goal-path?," the answer is affirmative. All of the reciprocities remain intact under the displacement of the equilibrium point (identity element) to a new goal on the generalized manifold. (The proof is not trivial because of the bracketed terms of G that are operators which, taken separately, are neither zero nor constants—except in the null case where the length of the goal-path is $\omega = 0$ so that $C = C'$ at both the initial and final condition. This would be to intend the goal of remaining in your current resting state).

This argument alone does not prove that this group of conserved variables is necessarily associated with the conserved quantity of interest. We address this issue next.

22.6.3 Second proof: The variables of G commute with the generalized Hamiltonian

Let α and β represent dynamical variables of G and G the generalized Hamiltonian. The Lie bracket representation of the perceiving-acting cycle implies that we must have

$$[\alpha, G] + [G, \beta] = \left. -\frac{\partial \alpha}{\partial t} \right|_{\omega} \left. -\frac{\partial \beta}{\partial t} \right|_{-\omega}$$

$$[G, \beta] + [\alpha, G] = \left. -\frac{\partial \beta}{\partial t} \right|_{\omega} \left. -\frac{\partial \alpha}{\partial t} \right|_{-\omega}$$

$$[\beta, G] + [G, \alpha] = \left. -\frac{\partial \beta}{\partial t} \right|_{\omega} \left. -\frac{\partial \alpha}{\partial t} \right|_{-\omega}$$

$$[G, \alpha] + [\beta, G] = \left. -\frac{\partial \alpha}{\partial t} \right|_{\omega} \left. -\frac{\partial \beta}{\partial t} \right|_{-\omega}$$

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[10] Normally, a complex Lie algebra may be the complexification of more than one real Lie algebra. This permits the possibility that the restricted real algebra that we have selected to represent the perceiving-acting cycle is not unique; other representations might exist. However, for all the real forms that a given complex simple Lie algebra might have, there is precisely one, the *compact real form*, which is the real Lie algebra obtained from a compact Lie group. A real semisimple Lie algebra is compact if and only if its bilinear Killing form is negative definite (Belinfante & Kalman, 1972, p. 81).

(Here we assume that we have divided through by i). If these quantities are identically zero, then both α and β are invariants of motion. Now let intention be one of the two variables, say, $\beta = B = -i$. Recall that

$$\text{if } i[G, \beta] + \partial\beta/\partial t = 0, \text{ then } -i[G, \beta] = i[\beta, G] = \partial\beta/\partial t \quad (22.13)$$

Therefore if we can prove that given $-i[G, B] = i[B, G] = 0$, it follows that $\partial B/\partial t = 0$ then we would have shown that intention, B, as a dynamical variable, both commutes with G, and as a consequence is a *new motion invariant* as sought. Consider

$$-i[G, B] = -[G, iB] = -[G, C] = -[G, 1] = -G(1) + (1)G = 0$$

Therefore, $\partial B/\partial t = 0$. This result can be obtained with any of the other terms of $G(A, B, C, D)$ in a similar manner so that, in general, for $\alpha = \{A, B, C, D\}$, $\partial\alpha/\partial t = 0$ is true. It is also true that the Poisson bracket of G with any two invariants of motion, α, β, is also an invariant of motion (Poisson's theorem). Using the Jacobi identity, this theorem is expressed as

$$-i[G, [a, b]] = 0 \quad (22.14)$$

As before, let $\alpha(P, P^*, Q, Q^*)$ and $\beta(P, P^*, Q, Q^*)$ stand, respectively, for the time-forward and time-backward components of total useful energy and total control information, Thus we arrive at the not too surprising conclusion that *the intentional flow of a generalized potential is conserved (satisfied) if and only if the product taking the difference of the inverted products of the total energy and total information should vanish.*

22.6.4   Third proof: Intention as the generator of G

We have asserted that intention ($B = -i$) is unique among the dynamical variables comprising G because it sets up the algebra of G and binds all of the other dynamical variables comprising G to the stationary goal-path of a conserved quantity—total action. Let us see why this is the case.

Recall that the complexification of G, (i.e., G^C) consists of two subgroups, g_1 and ig_2, originally identified as the real subgroup and the imaginary subgroup, respectively. Through the real restriction on G^C we obtained the isomorphic compact simple Lie group G^R possessing a pair of real subgroups coupled by i—construed as a real operator. Hence we arrived at

$$G = g_1 \bigcup ig_2 = [(1, 0), (-1, 0)], [(0, -1), (0, 1)] = [(C, A), (B, D)] \quad (22.1)$$

In addition, we saw that the generator for $G(A, B, C, D)$ could not be the detection operator, $A = -1$, or the control operator, $C = 1$, for neither of these generate the other members of G, $B = -i$ and $D = i$. Clearly, however, $A = -1$ is the generator for the real subgroup $g_1 = (C, A) = (1, -1)$, and not $C = 1$ because $-1^2 = 1$ while $1^2 = 1$. Similarly, it is just as clear that the generator for the real subgroup $g_2 = ig_1 = i(-1, 1) = (-i, i)$ is the complexified intention operator $iB = -1$, that is, $B = -i$, and can not be the complexified goal operator $iD = 1$, that is, $D = i$ for the same reason.

Although either i^n or $-i^n$ could be the generator for G if we ignored the real restriction on G, we are not at liberty to do so. Because of the real restriction on G, we must treat both B and D as real-valued operators within their subgroup that may only be related to the members of the other reciprocal subgroup through i. They may not, however, relate to each other through i, for i binds g_1 to g_2 and vice-versa, but does not bind members within a subgroup to each other. This proves the uniqueness of intention as an operator.

Intuitively, this argument asserts that G must satisfy the initial conditions as follows: Intention B must apply first to select a goal in terms of target and manner parameters so that its mode of control might be determined. Selection of the goal D makes available information to be detected by A, which then reciprocates with control operator C to determine the behavioral output. Here we see that $B = -i$ acts as the time-backward anticipatory flow operator to bind A to C with respect to goal D. Dually, $D = i$ acts as the time-forward causal flow operator to bind C to A with respect to intention B. This follows from the obvious fact that goals must be anticipated before they can be pursued.